PASSIVE FIELDS AND PARTICLES IN CHAOTIC FLOWS

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Abstract Two examples for the interplay between chaotic dynamics and stochastic forces within hydrodynamical systems are considered. The first case concerns the relaxation to equilibrium of a concentration field subject to both chaotic advection and molecular diffusion. The concentration field develops filamentary structures and the decay rate depends non-monotonically on the diffusion strength. The second example concerns polymers, modelled as particles with an internal degree of freedom, in a chaotic flow. The length distribution of the polymers turns out to follow a power law with an exponent that depends on the difference between Lyapunov exponent and internal relaxation rate.

Keywords: Chaotic advection, power law distribution, surface flows, polymers

Introduction

Fluid dynamics provides many prime examples for nonlinear dynamical systems. The origin of the nonlinearity is the advection of particles and fields by the underlying velocity field and takes, with $\mathbf{x}_P(t)$ the position of a particle, the form

$$\dot{\mathbf{x}}_P(t) = \mathbf{u}(\mathbf{x}_P(t), t)\,. \tag{1}$$

We assume here and in the following that the particles are neutrally buoyant so that they follow the instantaneous velocity field (for other particles, see Maxey and Riley, 1983). Typically, the fields are not constant or linear, so that (1) is a system of nonlinear equations and the dynamics of the particles becomes chaotic (Aref, 1984; Ottino, 1989; Acrivos et al., 1991; Aref, 1994; Aref, 2002). Stochastic elements enter if in addition to the deterministic parts also random forces, such as

molecular diffusion or Brownian motion, have to be taken into account. Further complications arise when the particles can react back onto the flow (e.g. by a density contrast or by modifying viscosity). They then become dynamically active. In the situations considered below no such feedback is allowed and the particles remain dynamically passive.

Here we consider two examples of the interplay between nonlinear advection and stochastic forcing: the relaxation of scalar fields in experiments with 2-d flows (Rothstein et al., 1999; Williams et al., 1997; Jütner et al., 1997) and the stretching of polymers in chaotic flows. In all cases the flow fields have a simple spatial structure and at most a periodic time dependence, i.e. we have deterministic chaos. If these deterministic flow fields are replaced by temporally fluctuating ones with prescribed spatial correlations techniques from stochastic analysis can be used to characterize the particle dynamics further (Falkovich et al., 2002; Klyatskin et al., 1996; Cranston and Scheutzow, 2002). However, both deterministic and random flows show similar behaviour.

In the first example we focus on the processes that contribute to the relaxation of a non-uniform concentration, e.g. a blob of dye, to a uniformly spread out concentration in a chaotic flow. By its chaotic nature the flow by itself will stir the blob into an object with filaments that grow longer and finer and eventually cover space. Diffusion alone will also result in a uniform spreading in space, albeit on a different time scale. When combined, these processes do not simply add up but give rise to non-trivial modulations in the relaxation rate.

In the second example we focus on the effects of the velocity gradients on particles with internal degrees of freedom, specifically long flexible polymers. The phenomenon of turbulent drag reduction (Lumley, 1969; Virk, 1975) is connected with the uncoiling of the polymers in the flow. Once uncoiled, the polymers are no longer passive but begin to change the velocity field, so that a reduced drag is observed. The passive dynamics of polymers in a prescribed flow field then is a first step towards a description of the polymer dynamics in a turbulent flow. It is also a nice example of how the presence of both additive and multiplicative fluctuating forces can give rise to a power law probability distribution (Balkovsky et al., 2000; Balkovsky et al., 2001; Chertkov, 2000; Sornette and Cont, 1997).

Relaxation in chaotic advection

Since the paper by Aref (Aref, 1984), in which the term chaotic advection was coined, many theoretical and experimental studies have addressed various aspects of the phenomenon (Ottino, 1989; Acrivos et al.,

1991; Aref, 1994; Williams et al., 1997; Rothstein et al., 1999). However, in order to arrive at mixing on a molecular level, diffusion has to be added in. Already in (Eckart, 1948) the differences between stirring and mixing are emphasized: during stirring the fluid elements are stretched and folded, but they always keep their identity. Mixing requires molecular diffusion to cause a diffusive exchange of markers between different fluid elements and to thus blur distinguishability of the origin of the fluid elements. It is obvious that stirring enhances mixing, but there will be mixing even without stirring. The persistent patterns proposed in (Pierrehumbert, 1994) and observed experimentally in (Rothstein et al., 1999) and the studies by (Fereday et al., 2002) highlight some of the effects that arise when chaos and diffusion interact to turn stirring into mixing.

For fast processes the effects of diffusion can be neglected. Typical times for the stirring by the flow are set by L/U_0, where U_0 and L are characteristic velocity and length scales, respectively. Diffusive spreading happens on a time scale L^2/D, where D is the diffusion constant, and is usually much slower. The experiments by Taylor on reversibility at low Reynolds number (Taylor, 1960; Homsy et al., 2000) also indicate this vast separation of time scales: the flow is reversed before diffusion has time to destroy the memory of the initial conditions (a further discussion of such echo phenomena in chaotic flows is given in (Eckhardt, 2003)).

However, this estimate has to be reconsidered when the dynamics builds up finer and finer structures and increasing gradients so that diffusion can no longer be neglected (cf. Pierrehumbert, 1994; Fereday et al., 2002). Consider particle spreading in a 2-d flow designed to describe the experiments of (Williams et al., 1997; Rothstein et al., 1999). The stream function is (Eckhardt and Hascoët, 2002)

$$\begin{aligned}\psi(x,y,t) &= \frac{f_0}{3\sqrt{9+\omega^2}}\sin(\omega t-\delta_1)\sin x\,\cos\sqrt{2}y \\ &\quad +\frac{5f_0}{27\sqrt{729+\omega^2}}\sin(\omega t-\delta_2)\sin 5x\,\cos\sqrt{2}y\,. \qquad (2)\end{aligned}$$

where f_0 is the amplitude of the periodic forcing which sets the velocity scale. The two modes respond to the driving with the phase delays $\delta_1 = \mathrm{atan}(\omega/3)$ and $\delta_2 = \mathrm{atan}(\omega/27)$, respectively, the difference in δ_1 and δ_2 being a prerequisite for chaotic motion. Fig. 1 shows a Poincaré surface of section and Fig. 2 a sequence of stroboscopic plots for many particles started in a small area. The initial distribution is stretched out along the unstable manifolds nearby and contracts along the stable

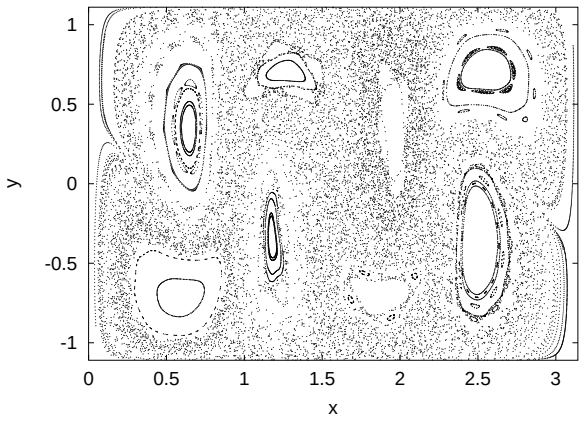

Figure 1. Stroboscopic map for the flow (2). The map was obtained by iterating 100 uniformly distributed initial conditions for 200 periods each and plotting all intermediate points. The frequency of the forcing is $\omega = 20$ and its amplitude $f_0 = 300$.

ones. It soon folds back just as the manifolds do. After a few periods the particles are more or less uniformly distributed in the chaotic region.

The presence of islands in the flow adds a slow component due to the trapping near islands. The relaxation becomes even slower when diffusion is added since stochastic perturbations can bring particles into islands from which they can escape with difficulty only. Such localization in islands is reflected in eigenvalues and eigenmodes (O. Popovich and A. Pikovsky, in preparation).

But even if there are no islands diffusion can have a nontrivial influence on the decay rate. In order to be able to vary the diffusion constant over a wide range and still be numerically reliable we illustrate

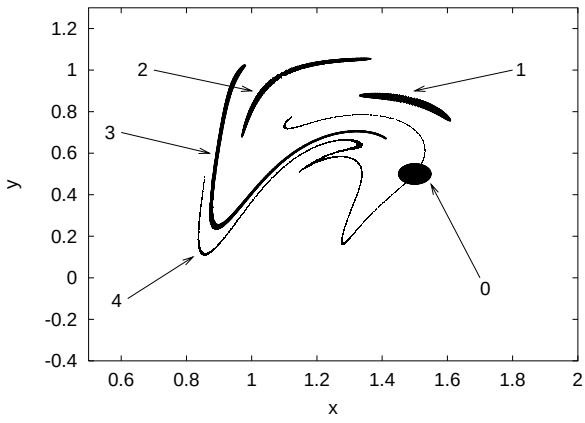

Figure 2. Evolution of a small blob of ten thousand particles after a few periods. Labels indicate multiples of a period: $t = nT$ with $n = 0$ (initial configuration) and $n = 1$, 2, 3 and 4, after which time one tendril of the stretched out blob overlaps with the initial configuration. The parameters are frequency $\omega = 40$ and the forcing amplitude $f_0 = 1500$.

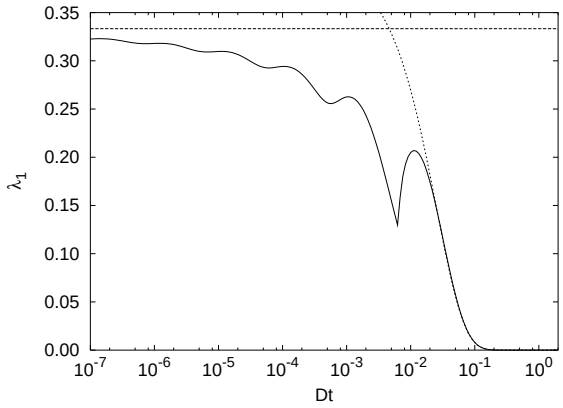

Figure 3 Variation of leading eigenvalue with diffusion for the map (3). The calculated eigenvalue is indicated by the thick line. For extremely small diffusion it approaches the eigenvalue of the deterministic map (dashed horizontal line), and for large diffusion the slowest one of the unperturbed diffusion equation (dotted line).

this possibility for 1-d maps on a ring, e.g.

$$f(x) = \begin{cases} 3x & 0 < x < 1/3 \\ 2 - 3x & 1/3 < x < 2/3 \\ -2 + 3x & 2/3 < x < 1 \end{cases} . \tag{3}$$

The map is periodically continued and noise is added to simulate molecular diffusion. The calculation proceeds by taking the Frobenius-Perron equation for the first iteration step, followed by a Gaussian smearing. The eigenvalues can be calculated by numerical diagonalization in either position space or Fourier space (Hascoët and Eckhardt, to be published).

The spectrum of the unperturbed Frobenius Perron equation can be calculated e.g. from periodic orbit theory. The eigenvalue next to 1 comes out to be 1/3. Similarly, if diffusion is large enough, the deterministic part is a small perturbation and we expect eigenvalues that depend on diffusion like $\exp(-4\pi^2 Dt)$, where t is the time interval over which diffusion acts in-between iterations of the map. The leading eigenvalue of the spectrum is shown in Fig. 3. Instead of a simple cross over from fast chaotic relaxation for small diffusion to diffusional decay for large D we see non-monotonic variations with intriguing oscillations.

Similar non-monotonic behaviour has been observed in diffusion in chains of tent maps when the height of the tip of the tent was modified (Klages, 2002; Klages and Dorfman, 1999; Klages and Dorfman, 1997; Klages and Dorfman, 1995). There, the non-monotonic decay could be traced back to the non-monotonic changes in the phase space structures, the homoclinic and heteroclinic tangles and the changes in the efficiency that particles could be exchanged with between different sets of the Markov partition. We expect a similar mechanism here: the decay within the classical dynamics is connected with the efficient

stretching and folding. Diffusion can help by mixing faster within cells, but it can also slow down relaxation by transporting fluid into regions with less efficient mixing.

Polymer stretching in chaotic flows

The second example we consider is the dynamics of long flexible polymers (Lumley, 1969; Virk, 1975). In fluids at rest these polymers are coiled up into tiny spheres. However, despite their small size, they can interact with gradients of the flow field in turbulent and also in chaotic flows. When the strains are stronger than their entropic relaxation forces they can be stretched out to form elongated structures or even thin threads. Elongated polymers influence the viscosity and change the dynamics on the smallest scales. In the case of turbulent drag reduction these microscopic modifications add up to large scale changes in the velocity field that result in noticeable reductions in drag (Lumley, 1969; Virk, 1975). As a part of a program to understand the effects of polymers on the flow we need to understand the size distribution of the polymers.

In the simplest, and for our purposes sufficient model for the polymer, the large number of monomers and flexible elements are replaced by their inertial tensor, which can be written as a dyad formed by a vector $\mathbf{R}$, a kind of end-to-end distance vector (Bird et al., 1987). This vector is driven by thermal fluctuations ξ, modelled as a 3-D Gaussian process. The entropic preference of the molecule for a coiled state results in an entropic restoring force, which contributes a relaxation time λ towards the equilibrium configuration. The small size of the polymers allows to linearize the flow field, so that only local gradients enter. We thus have a polymer transported along a Lagrangian path $\mathbf{x}_P(t)$, with an internal dynamics described by

$$\dot{R}_i = \left(\frac{\partial u_i}{\partial x_j}(t) - \frac{1}{2\lambda} \right) R_j + \xi_i \,. \tag{4}$$

From this vector one can form the configuration tensor, $c_{ij} = \langle R_i R_j \rangle$, where the averages are over thermal noise. The quantity used to measure extension of the polymers is $\mathrm{tr}\, c$. Scales can be fixed so that in equilibrium $c_{ij} = \delta_{ij}$ and $\mathrm{tr}\, c = 3$.

Since the polymers are advected by the flow the local gradients vary. Asymptotically in the infinite time average they approach the Lyapunov exponent of the trajectory. In order to capture some of the variations of the local stretching rate over times compatible with the internal relaxation time of the polymers, we consider distributions of finite time

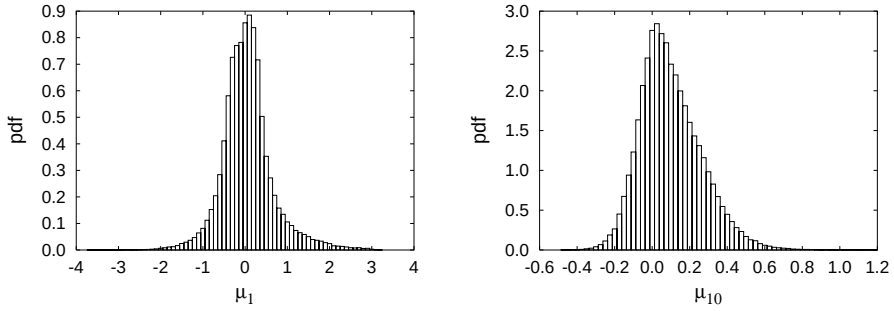

Figure 4. Finite time Lyapunov exponents for two different time intervals T and the velocity field (7). The mean in both cases is $\bar{\mu} = 0.11$, and the empirical variance decreases from 0.373 for $T = 1$ to 0.026 for $T = 10$.

Lyapunov exponents,

$$\mu_T = \frac{1}{T} \log \frac{\|\delta \mathbf{x}(t_i + T)\|}{\|\delta \mathbf{x}(t_i)\|} \,. \tag{5}$$

They measure the growth rate of a separation $\delta \mathbf{x}$ over a time interval T. The equation for $\delta \mathbf{x}$ is the variational equation

$$\delta \dot{x}_i = \frac{\partial u_i}{\partial x_j}(t) \delta x_j \,. \tag{6}$$

As the time T increases the distribution narrows around the infinite time Lyapunov exponent (Fig. 4).

The length distribution of the polymers results from a balance between the stretching by the gradients of the velocity field, as measured by the Lyapunov exponent μ, and the entropic relaxation. If the gradients and thus the Lyapunov exponents were constant, the thermal fluctuations would give rise to a Gaussian distribution. However, in the presence of fluctuations in the Lyapunov exponent a power law distribution of the size of the polymers follows (Balkovsky et al., 2000; Balkovsky et al., 2001; Chertkov, 2000). A corresponding result in a more general context of multiplicative stochastic processes with a reflecting barrier is given in (Sornette and Cont, 1997). In fact, the noise term in (4) can be replaced by a reflecting barrier without changing the tails of the polymer length distribution. Evidently, in a power law distribution large extensions of the polymer are much more likely than in a Gaussian distribution and thus this higher probability results in a more effective and efficient back reaction of the polymers on the flow.

Since it makes no difference for the polymer dynamics if the stretching results from a spatially and temporally fluctuating turbulent flow or

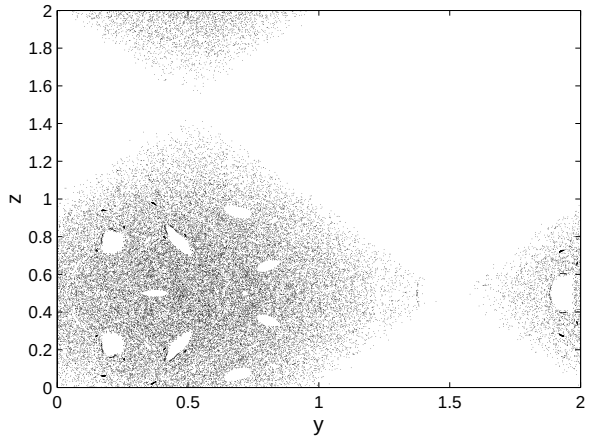

Figure 5 Poincaré surface of section for flow (7) with parameters $A = B = 1$, $C = 0.3$ at $x = 0.5$. The non-uniform distibution of points is connected with a non-uniform invariant density. The holes indicate elliptic islands.

from a stationary flow with chaotic trajectories, we can also analyze the dynamics of polymers in a 3-d incompressible flow with chaotic trajectories, i.e.

$$\mathbf{u} = \begin{pmatrix} (A\sin\pi y + B\sin\pi z)\sin\pi x \\ A\cos\pi x\cos\pi y + C\cos\pi z \\ B\cos\pi x\cos\pi z + C\sin\pi y \end{pmatrix} \tag{7}$$

This flow is similar to the ABC flows (Dombre et al., 1986) but satisfies free slip boundary conditions in the planes $z = 0$ and $z = 1$. The particle trajectories are trapped between the surfaces, as they would be in a planar shear flow. Fig. 5 shows a Poincaré surface of section of a chaotic trajectory in this flow. The distribution of finite time Lyapunov exponents (Fig. 4) indicates a mean Lyapunov exponent of about 0.11.

As a result of the interplay between the exponential stretching and its fluctuations one ends up with a power law distribution (Fig. 6) with an exponent that depends on the relaxation rate λ. The distribution loses its normalizability (exponent -1 or larger) when $2\mu\lambda = 1$. Numerical calculations for the polymer length support this (Fig. 7).

The power law distribution has also been identified in turbulent flows (Eckhardt et al., 2002). The consequences for the velocity field when the backreaction of the polymers to the flow is allowed for are under investigation.

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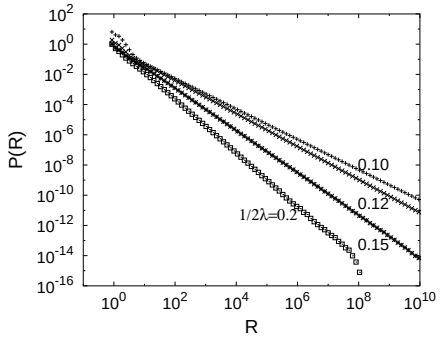

Figure 6. Polymer length distribution (not normalized) for different values of the polymeric relaxation time λ.

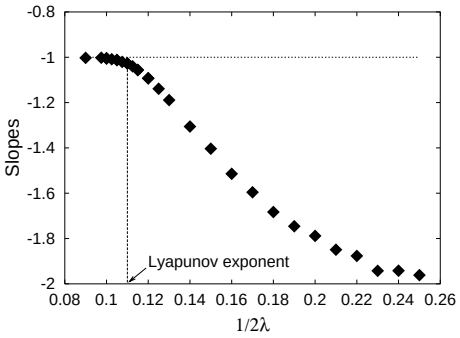

Figure 7. Exponent in the distribution of $tr\, c$ vs. the inverse of the relaxation time λ.

References

Acrivos, A., Aref, H., and Ottino, J. M., editors (1991). *Symposium on Fluid Mechanics of Stirring and Mixing*, volume 5 of *Phys. Fluids A*.

Aref, H. (1984). Stirring by chaotic advection. *J. Fluid. Mech.*, 143:1.

Aref, H., editor (1994). *Chaos Applied to Fluid Mixing*, volume 4 of *Chaos, Solitons and Fractals*.

Aref, H. (2002). The development of chaotic advection. *Phys. Fluids*, 14:1315.

Balkovsky, E., Fouxon, A., and Lebedev, V. (2000). Turbulent dynamics of polymer solutions. *Phys. Rev. Lett.*, 84:4765–4768.

Balkovsky, E., Fouxon, A., and Lebedev, V. (2001). Turbulence of polymer solutions. *Phys. Rev. E*, 64:056301.

Bird, R. B., Armstrong, R. C., and Hassager, O. (1987). *Dynamics of polymeric liquids*, volume I. (Fluid mechanics), II. (Kinetic theory). John Wiley & Sons, New York.

Chertkov, M. (2000). Polymer stretching by turbulence. *Phys. Rev. Lett.*, 84:4761.

Cranston, M. and Scheutzow, M. (2002). Dispersion rates under finite mode kolmogorov flows. *Ann. Appl. Prob.*, 12:511.

Dombre, T., Frisch, U., Greene, J. M., Henon, M., Mehr, A., and Soward, A. M. (1986). Chaotic streamlines in the ABC flows. *J. Fluid. Mech.*, 167:353–391.

Eckart, C. (1948). An analysis of the stirring and mixing processes in incompressible fluids. *J. Mar. Res.*, 7:265.

Eckhardt, B. (2003). Echoes in classical dynamical systems. *J. Phys. A*, 36:371.

Eckhardt, B. and Hascoët, E. (2002). Chaotic advection by viscous dephasing. *preprint*.

Eckhardt, B., Kronjäger, J., and Schumacher, J. (2002). Stretching of polymers in a turbulent environment. *Comp. Phys. Commun.*, 147:538–543.

Falkovich, G., Gawedzki, K., and Vergassola, M. (2002). Particles and fields in fluid turbulence. *Rev. Mod. Phys.*, 73:913.

Fereday, D. R., Haynes, P. H., Wonhas, A., and Vassilicos, J. C. (2002). Scalar variance decay in chaotic advection and batchelor-regime turbulence. *Phys. Rev. E*, 65:35301.

Homsy, G. M., Aref, H., Breuer, K. S., Hochgreb, S., Koseff, J. R., Munson, B. R., and Powell, K. G. (2000). Multi-media fluid mechanics. *CD-ROM Cambridge University Press*.

Jütner, B., Marteau, D., Tabeling, P., and Thess, A. (1997). Numerical simulations of experiments on quasi-two-dimensional turbulence. *Phys. Rev. E*, 55:5479.

Klages, R. (2002). Transitions from deterministic to stochastic diffusion. *Europhys. Lett.*, 57:796.

Klages, R. and Dorfman, J. (1995). Simple maps with fractal diffusion coefficients. *Phys. Rev. Lett.*, 74:387.

Klages, R. and Dorfman, J. (1999). Simple deterministic dynamical systems with fractal diffusion coefficients. *Phys. Rev. E.*, 59:5361.

Klages, R. and Dorfman, J. R. (1997). Dynamical crossover in deterministic diffusion. *Phys. Rev. E.*, 55:1247.

Klyatskin, V. I., Woyczynski, W. A., and Gurarie, D. (1996). Diffusing passive tracers in random imcompressible flows: statistical topography aspect. *J. Stat. Phys.*, 84:797.

Kronjäger, J., Braun, W., and Eckhardt, B. (2002). Polymer stretching in stationary flows. *preprint*.

Lumley, J. L. (1969). Drag reduction by additives. *Ann. Rev. Fluid Mech.*, 1:367.

Maxey, M. R. and Riley, J. J. (1983). Equation of motion for small rigid sphere in a nonuniform flow. *Phys. Fluids*, 26:883.

Ottino, J. M. (1989). *The kinematics of Mixing : Stretching, Chaos and Transport*. Cambridge University Press.

Pierrehumbert, R. (1994). Tracer microstructure in the large-eddy dominated regime. *Chaos, Solitons, Fractals*, 4:347.

Rothstein, D., Henry, E., and Gollub, J. P. (1999). Persistent patterns in transient chaotic mixing. *Nature*, 401:770–772.

Sornette, D. and Cont, R. (1997). Convergent multiplicative processes repelled from zero: Power laws and truncated power laws. *J. Phys. I France*, 7:431–444.

Taylor, G. (1960). Low reynolds numbers flow. *(16mm film).(Newton, Massachusetts: Educational Services Incorporated.)*.

Virk, P. S. (1975). Drag reduction fundamentals. *AIChE J.*, 21:625–656.

Williams, B. S., Marteau, D., and Gollub, J. P. (1997). Mixing of a passive scalar in magnetically forced two-dimensional turbulence. *Phys. Fluids*, 9:2061–2080.